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Exact evaluation of the Baxter–Bazhanov Green function

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Abstract. The analytic properties of the lattice Green function

$$G(t) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{t - \cos x - \cos y - \cos z + \cos x \cos y \cos z},$$

where t lies in a complex plane which is cut along the real axis from -2 to $+2$, are investigated. In particular, it is proved that $tG(t)$ can be written in the product form

$$tG(t) = (1-\xi^2)^{-\frac{1}{4}}\left[{}_2F_1\left(\frac{1}{8}, \frac{5}{8}; 1; \xi^2\right)\right]^2,$$

where $\xi = 2/t$ and ${}_2F_1(a, b; c; z)$ denotes a hypergeometric function. This result and the analytic continuation formulae for the ${}_2F_1$ function are then used to obtain various exact closed-form expressions for the related functions

$$G_{\mathrm{R}}(s) = \mathrm{Re}\left[\lim_{\epsilon\to 0+} G(s - \mathrm{i}\epsilon)\right],$$

$$G_{\mathrm{I}}(s) = \mathrm{Im}\left[\lim_{\epsilon\to 0+} G(s - \mathrm{i}\epsilon)\right],$$

where $s \in (-2, 2)$. It is also shown that $G(t)$ is a solution of a third-order Fuchsian differential equation.

1. Introduction

Recently, Baxter and Bazhanov (Glasser, private communication) have shown that the triple integral

$$I(t) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \ln(t - \cos x - \cos y - \cos z + \cos x \cos y \cos z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad (1.1)$$

can be evaluated exactly for the special case $t = 2$. In particular, they found that

$$I(2) = \frac{8}{\pi}G - 3\ln 2, \qquad (1.2)$$

where G is the Catalan constant. An alternative more direct derivation of this result has also been given by Glasser (unpublished work).

Our main aim in this paper is to investigate the analytic properties of the derivative function

$$G(t) \equiv I'(t) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{t - \cos x - \cos y - \cos z + \cos x \cos y \cos z}. \qquad (1.3)$$

Integral (1.3) defines a single-valued analytic function $G(t)$ in the complex t plane provided that a cut is made along the real axis from -2 to $+2$. It is also clear that the function $G(t)$ is a form of lattice Green function (see Katsura *et al* 1971). A series representation for

$G(t)$ can be derived by expanding the integrand in (1.3) in inverse powers of t and then integrating term by term. This procedure gives

$$G(t) = \frac{1}{t}\sum_{k=0}^{\infty}\frac{\mu_k}{t^k}, \qquad |t| > 2 \tag{1.4}$$

where

$$\mu_k = \frac{1}{\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}(\cos x + \cos y + \cos z - \cos x\cos y\cos z)^k\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{1.5}$$

From (1.5) one finds that $\mu_0 = 1$, $\mu_2 = \frac{13}{8}$ and $\mu_4 = \frac{2235}{512}$. It is also readily seen that the odd coefficients $\{\mu_{2n+1}; n = 0, 1, 2, \ldots\}$ are equal to zero.

In many physical applications of Green functions one requires the limiting behaviour of $G(t)$ as the complex variable t approaches the real axis. It is convenient, therefore, to introduce the additional definitions

$$G^{\pm}(s) \equiv \lim_{\epsilon\to 0+} G(s \pm \mathrm{i}\epsilon) \equiv G_{\mathrm{R}}(s) \mp \mathrm{i}G_{\mathrm{I}}(s), \tag{1.6}$$

where $-\infty < s < \infty$. The imaginary part $G_{\mathrm{I}}(s)$ can be used to express the Green function $G(t)$ in the alternative simplified form (Katsura *et al* 1971)

$$G(t) = \int_{-\infty}^{\infty}\frac{\rho(s)}{t-s}\,\mathrm{d}s, \tag{1.7}$$

where the weight function

$$\rho(s) = \frac{1}{\pi}G_{\mathrm{I}}(s). \tag{1.8}$$

It follows from (1.4) and (1.7) that

$$\mu_k = \int_{-\infty}^{\infty} s^k\rho(s)\,\mathrm{d}s, \tag{1.9}$$

where $k = 0, 1, 2, \ldots$.

2. Basic results

We begin by performing the integration over the variable z in (1.3). This procedure gives

$$G(t) = \frac{1}{\pi^2}\int_0^{\pi}\frac{\mathrm{d}x}{\sin x}\int_0^{\pi}\frac{\mathrm{d}y}{\sqrt{(a-\cos y)(b-\cos y)}}, \tag{2.1}$$

where

$$a = (t - \cos x + 1)/(1 + \cos x), \tag{2.2}$$
$$b = (t - \cos x - 1)/(1 - \cos x). \tag{2.3}$$

We now make the substitution $u = \cos y$ in (2.1) and apply the standard result (Byrd and Friedman 1971, p 107)

$$\int_{-1}^{1}\frac{\mathrm{d}u}{\sqrt{(a-u)(b-u)(1-u^2)}} = \frac{2}{\sqrt{(a-1)(b+1)}}K(k), \tag{2.4}$$

where

$$k^2 = \frac{2(a-b)}{(a-1)(b+1)} \tag{2.5}$$

and $K(k)$ is a complete elliptic integral of the first kind. Hence, we find that

$$G(t) = \frac{\xi}{\pi^2}\int_0^{\pi} \frac{1}{1-\xi\cos x} K(k)\,\mathrm{d}x, \tag{2.6}$$

where

$$k^2 = 1 - \frac{1-\xi^2}{(1-\xi\cos x)^2} \tag{2.7}$$

and $\xi = 2/t$.

In order to evaluate the integral (2.6) we first expand the integrand in powers of k^2 using the hypergeometric series

$$\frac{2}{\pi}K(k) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{(1)_n^2} k^{2n}, \qquad |k^2| < 1 \tag{2.8}$$

where $(a)_n$ denotes a Pochhammer symbol, and then apply the binomial theorem to the expression for k^{2n}. This procedure yields

$$tG(t) = \sum_{n,m=0}^{\infty} \frac{(\frac{1}{2})_n^2(-n)_m}{(1)_n^2(1)_m}(1-\xi^2)^m J_m(\xi), \tag{2.9}$$

where

$$J_m(\xi) = \frac{1}{\pi}\int_0^{\pi} \frac{\mathrm{d}x}{(1-\xi\cos x)^{2m+1}}. \tag{2.10}$$

It is known that (Gradshteyn and Ryzhik 1980, p 383)

$$J_m(\xi) = \frac{1}{(1-\xi^2)^{m+\frac{1}{2}}} P_{2m}\left(\frac{1}{\sqrt{1-\xi^2}}\right), \tag{2.11}$$

where $P_\nu(z)$ denotes a Legendre polynomial. We can use the standard formula

$$P_\nu(z) = {}_2F_1(-\nu, \nu+1; 1; \tfrac{1}{2} - \tfrac{1}{2}z) \tag{2.12}$$

to write (2.11) in the form

$$J_m(\xi) = (1-\xi^2)^{-m-\frac{1}{2}}\,{}_2F_1(-2m, 2m+1; 1; \omega), \tag{2.13}$$

where

$$\omega = \tfrac{1}{2} - \tfrac{1}{2}(1-\xi^2)^{-\frac{1}{2}}. \tag{2.14}$$

The substitution of formula (2.13) into equation (2.9) gives

$$tG(t) = (1-\xi^2)^{-\frac{1}{2}} \sum_{n,m,j=0}^{\infty} \frac{(\frac{1}{2})_n^2(-n)_m(-2m)_j(2m+1)_j}{(1)_n^2(1)_m(1)_j^2}\omega^j, \tag{2.15}$$

provided that $|\omega| < 1$.

A simplification of the triple series in (2.15) can be achieved by applying the contour integral representation

$$(-2m)_j(2m+1)_j = (-1)^j \frac{(2j)!}{2\pi\mathrm{i}} \int_{\mathcal{C}_1} \frac{(1+z)^{2m+j}}{z^{1+2j}}\,\mathrm{d}z \tag{2.16}$$

to the summation, where $\mathcal{C}_1$ is a closed contour which encloses the origin $z = 0$ with a winding number $+1$. (It is also necessary to restrict $\mathcal{C}_1$ to lie in a z plane which is cut along the real axis from $-\infty$ to -1.) After carrying out the summations over m and n it is found that

$$tG(t) = (1-\xi^2)^{-\frac{1}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_j}{(1)_j} Q_j \omega^j, \tag{2.17}$$

where

$$Q_j = \frac{(-4)^j}{2\pi \mathrm{i}} \int_{\mathcal{C}_1} \frac{(1+z)^j}{z^{1+2j}} {}_2F_1\left[\frac{1}{2}, \frac{1}{2}; 1; -z(2+z)\right] \mathrm{d}z. \tag{2.18}$$

Next we apply the transformation formula (see Erdélyi *et al* 1953, p 113, equation (30))

$${}_2F_1\left[\frac{1}{2}, \frac{1}{2}; 1; -z(2+z)\right] = (1+z)^{-\frac{1}{2}} {}_2F_1\left[\frac{1}{2}, \frac{1}{2}; 1; -\frac{z^2}{4(1+z)}\right] \tag{2.19}$$

to equation (2.18). Hence we obtain

$$Q_j = (-4)^j \sum_{m=0}^{j} \frac{\left(\frac{1}{2}\right)_m^2}{(1)_m^2} \left(-\frac{1}{4}\right)^m \mathrm{Res}(j, m; 0), \tag{2.20}$$

where $\mathrm{Res}(j, m; 0)$ is the residue of the function

$$f(j, m; z) = z^{2m-2j-1}(1+z)^{j-m-\frac{1}{2}} \tag{2.21}$$

at the origin $z = 0$. It can be readily shown that

$$\mathrm{Res}(j, m; 0) = \frac{\left(\frac{1}{2}\right)_j (-j)_m}{(1)_j \left(-j+\frac{1}{2}\right)_m} (-4)^{m-j}. \tag{2.22}$$

The substitution of formula (2.22) into equation (2.20) then gives

$$Q_j = \frac{\left(\frac{1}{2}\right)_j}{(1)_j} {}_3F_2\left[\begin{matrix} -j, & \frac{1}{2}, & \frac{1}{2}; \\ -j+\frac{1}{2}, & 1; & \end{matrix} \; 1\right], \tag{2.23}$$

where ${}_3F_2$ denotes a generalized hypergeometric function.

We now apply the special hypergeometric identity (see Slater 1966, p 76)

$${}_3F_2\left[\begin{matrix} -j, & \frac{1}{2}, & \frac{1}{2}; \\ -j+\frac{1}{2}, & 1; & \end{matrix} \; 1\right] = \frac{\left(\frac{1}{4}\right)_j \left(\frac{3}{4}\right)_j}{\left(\frac{1}{2}\right)_j^2} {}_4F_3\left[\begin{matrix} -j, & -j, & \frac{1}{4}, & \frac{3}{4}; \\ -j+\frac{1}{4}, & -j+\frac{3}{4}, & 1; & \end{matrix} \; 1\right] \tag{2.24}$$

to equation (2.23). Hence we obtain

$$Q_j = \frac{\left(\frac{1}{4}\right)_j \left(\frac{3}{4}\right)_j}{\left(\frac{1}{2}\right)_j (1)_j} {}_4F_3\left[\begin{matrix} -j, & -j, & \frac{1}{4}, & \frac{3}{4}; \\ -j+\frac{1}{4}, & -j+\frac{3}{4}, & 1; & \end{matrix} \; 1\right]. \tag{2.25}$$

In appendix A we derive result (2.25) by using an alternative *more direct* procedure which does not use identity (2.24). The substitution of (2.25) into (2.17) then gives

$$tG(t) = (1-\xi^2)^{-\frac{1}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{4}\right)_j \left(\frac{3}{4}\right)_j}{(1)_j^2} {}_4F_3\left[\begin{matrix} -j, & -j, & \frac{1}{4}, & \frac{3}{4}; \\ -j+\frac{1}{4}, & -j+\frac{3}{4}, & 1; & \end{matrix} \; 1\right] \omega^j, \tag{2.26}$$

where $|\omega| < 1$. The series in (2.26) can be readily expressed in the simple product form (see Erdélyi *et al* 1953, p 187, equation (14))

$$tG(t) = (1-\xi^2)^{-\frac{1}{2}} [{}_2F_1(\tfrac{1}{4}, \tfrac{3}{4}, 1, \omega)]^2, \tag{2.27}$$

where $\xi = 2/t$ and the variable ω is defined in equation (2.14).

3. Analytic properties of $G(t)$

In this section we shall use the basic result (2.27) to investigate the mathematical properties of $G(t)$.

3.1. Transformation formulae for $G(t)$

The application of the quadratic transformation formula (Erdélyi *et al* 1953, p 112, equation (16))

$$ {}_2F_1(\tfrac{1}{8},\tfrac{5}{8};1;\xi^2) = (1-\xi^2)^{-\frac{1}{8}}\,{}_2F_1[\tfrac{1}{4},\tfrac{3}{4};1;\tfrac{1}{2}-\tfrac{1}{2}(1-\xi^2)^{-\frac{1}{2}}] \tag{3.1} $$

to equation (2.27) gives

$$ tG(t) = (1-\xi^2)^{-\frac{1}{4}}[{}_2F_1(\tfrac{1}{8},\tfrac{5}{8};1;\xi^2)]^2, \tag{3.2} $$

where $\xi = 2/t$. This result and the analytic continuation formulae for the ${}_2F_1$ hypergeometric function enable one to calculate the numerical value of $G(t)$ at *any* point in the cut t plane. We can also use the Kummer transformation formula (Erdélyi *et al* 1953, p 105, equation (2))

$$ {}_2F_1(\tfrac{1}{8},\tfrac{5}{8};1;\xi^2) = (1-\xi^2)^{\frac{1}{4}}\,{}_2F_1(\tfrac{3}{8},\tfrac{7}{8};1;\xi^2) \tag{3.3} $$

to write (3.2) in the alternative product form

$$ tG(t) = {}_2F_1(\tfrac{1}{8},\tfrac{5}{8};1;\xi^2)\,{}_2F_1(\tfrac{3}{8},\tfrac{7}{8};1;\xi^2). \tag{3.4} $$

The behaviour of $G(t)$ in the neighbourhood of the singular points $\xi = \pm 1$ can be determined by applying a standard analytic continuation formula (Erdélyi *et al* 1953, p 108, equation (1)) to the ${}_2F_1$ function in (3.2). It is found that

$$ \begin{aligned} tG(t) = (1-\xi^2)^{-1/4}\Bigg\{ & \frac{[\Gamma(\frac{1}{4})]^2}{2^{3/4}\pi^{3/2}}\,{}_2F_1\left(\frac{1}{8},\frac{5}{8};\frac{3}{4};1-\xi^2\right) \\ & - \frac{2^{7/4}\pi^{1/2}}{[\Gamma(\frac{1}{4})]^2}(1-\xi^2)^{1/4}\,{}_2F_1\left(\frac{3}{8},\frac{7}{8};\frac{5}{4};1-\xi^2\right)\Bigg\}^2, \end{aligned} \tag{3.5} $$

where $|\arg(1-\xi^2)| < \pi$.

Next we apply the further transformation formula (Erdélyi *et al* 1953, p 112, equation (16))

$$ {}_2F_1(\tfrac{1}{4},\tfrac{3}{4};1;\omega) = (1-\omega)^{-\frac{1}{4}}\,{}_2F_1[\tfrac{1}{2},\tfrac{1}{2};1;\tfrac{1}{2}-\tfrac{1}{2}(1-\omega)^{-\frac{1}{2}}] \tag{3.6} $$

to equation (2.27). Hence we find that

$$ tG(t) = (1-\xi^2)^{-\frac{1}{4}}\left[\frac{1}{2}+\frac{1}{2}(1-\xi^2)^{\frac{1}{2}}\right]^{-\frac{1}{2}}\left[\frac{2}{\pi}K(k)\right]^2, \tag{3.7} $$

where

$$ k^2 = \tfrac{1}{2} - \tfrac{1}{2}(1-\xi^2)^{\frac{1}{4}}\left[\tfrac{1}{2}+\tfrac{1}{2}(1-\xi^2)^{\frac{1}{2}}\right]^{-\frac{1}{2}}. \tag{3.8} $$

It is interesting to note that results similar to (3.7) have also been obtained for the three cubic lattice Green functions (Watson 1939; Joyce 1971, 1994).

Finally, we can use the identity (Prudnikov *et al* 1990)

$$ \left[{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\omega\right)\right]^2 = {}_3F_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}, & \frac{1}{2}; \\ 1, & 1; & \end{matrix}\; 4\omega(1-\omega)\right] \tag{3.9} $$

to express (2.27) in the generalized hypergeometric form

$$tG(t) = (1-\xi^2)^{-\frac{1}{2}}\,{}_3F_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}, & \frac{1}{2}; & \\ & & & -\xi^2/(1-\xi^2) \\ 1, & 1; & & \end{matrix}\right], \tag{3.10}$$

where $\xi = 2/t$.

3.2. Differential equation for $G(t)$

We shall now use formula (3.10) to establish a Fuchsian differential equation for $G(t)$. It can be shown (Erdélyi *et al* 1953, p 184, equation (2)) that the generalized hypergeometric function

$$y(z) = {}_3F_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}, & \frac{1}{2}; & \\ & & & z \\ 1, & 1; & & \end{matrix}\right] \tag{3.11}$$

is a solution of the third-order differential equation

$$[\vartheta^3 - z(\vartheta+\tfrac{1}{4})(\vartheta+\tfrac{3}{4})(\vartheta+\tfrac{1}{2})]y = 0, \tag{3.12}$$

where $\vartheta = z(\mathrm{d}/\mathrm{d}z)$. From this result it follows that

$$32z^2(z-1)\frac{\mathrm{d}^3y}{\mathrm{d}z^3} - 48z(2-3z)\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + 2(51z-16)\frac{\mathrm{d}y}{\mathrm{d}z} + 3y = 0. \tag{3.13}$$

If we change the independent variable in (3.13) from z to $4/(4-t^2)$ and the dependent variable from y to $t[1-(4/t^2)]^{1/2}G$ then we obtain the required differential equation

$$(t^2-4)^2\frac{\mathrm{d}^3G}{\mathrm{d}t^3} + 6t(t^2-4)\frac{\mathrm{d}^2G}{\mathrm{d}t^2} + (7t^2-13)\frac{\mathrm{d}G}{\mathrm{d}t} + tG = 0. \tag{3.14}$$

The Riemann P-symbol (see Ince 1956, pp 370–2) associated with this Fuchsian equation is

$$P\left[\begin{matrix}-2, & +2, & \infty; & \\ 0, & 0, & 1; & t \\ -\frac{1}{4}, & -\frac{1}{4}, & 1; & \\ +\frac{1}{4}, & +\frac{1}{4}, & 1; & \end{matrix}\right]. \tag{3.15}$$

We see that the P-symbol (3.15) has the correct Fuchsian invariant of 3.

3.3. Properties of the even moments $\{\mu_{2n}; n = 0, 1, 2, \ldots\}$

The substitution of the series (1.4) in the differential equation (3.14) enables one to derive a recurrence relation for the non-zero moments $\{\mu_{2n}; n = 0, 1, 2, \ldots\}$. In particular, we find that

$$8(n+1)^3\mu_{2n+2} - (2n+1)(32n^2+32n+13)\mu_{2n} + 32n(4n^2-1)\mu_{2n-2} = 0, \tag{3.16}$$

where $n \geqslant 0$, with the initial conditions $\mu_0 = 1$ and $\mu_{-2} \equiv 0$. A closed-form expression for μ_{2n} can also be obtained by first expanding the right-hand side of equation (3.10) in powers of $\xi^2 = 4/t^2$. A comparison of this expansion with (1.4) then gives the simple formula

$$\frac{\mu_{2n}}{2^{2n}} = \frac{\left(\frac{1}{2}\right)_n}{(1)_n}\,{}_3F_2\left[\begin{matrix}-n, & \frac{1}{4}, & \frac{3}{4}; & \\ & & & 1 \\ 1, & 1; & & \end{matrix}\right]. \tag{3.17}$$

Unsuccessful attempts have been made to establish (3.17) by carrying out a *direct* evaluation of the triple integral (1.5).

Finally, we note that the asymptotic behaviour of μ_{2n} as $n \to \infty$ can be analysed by applying the method of Darboux (1878) to the analytic continuation (3.5). It is found that the asymptotic representation is

$$\frac{\mu_{2n}}{2^{2n}} \sim \left[\frac{\Gamma(\frac{1}{4})}{\pi\sqrt{2}}\right]^3 \frac{1}{n^{3/4}}\left[1 - \frac{1}{4n} + \frac{157}{3072n^2} - \frac{17}{12\,288n^3} - \frac{136\,631}{44\,040\,192n^4} - \frac{139\,409}{176\,160\,768n^5} + \frac{34\,671\,611}{23\,622\,320\,128n^6} + \cdots\right] - \frac{2}{\left[\Gamma\left(\frac{1}{4}\right)\right]^3}\frac{1}{n^{5/4}}\left[1 - \frac{1}{2n} + \frac{861}{5120n^2} - \frac{183}{10\,240n^3} - \frac{175\,167}{10\,485\,760n^4} + \frac{52\,731}{20\,971\,520n^5} + \frac{279\,788\,819}{27\,917\,287\,424n^6} + \cdots\right], \tag{3.18}$$

as $n \to \infty$. The numerical evaluation of (3.18) for the particular case $n = 25$ gives the approximation

$$\mu_{50}/2^{50} \approx 0.047\,388\,443\,365\,1630\ldots. \tag{3.19}$$

This value is in excellent agreement with the exact value

$$\mu_{50}/2^{50} = 0.047\,388\,443\,365\,1693\ldots. \tag{3.20}$$

4. Formulae for $G_{\mathrm{R}}(s)$ and $G_{\mathrm{I}}(s)$

We shall now determine the real part $G_{\mathrm{R}}(s)$ and the imaginary part $G_{\mathrm{I}}(s)$ of the Green function $G(s - \mathrm{i}\epsilon)$ as $\epsilon \to 0+$, where s is a fixed real number in the interval $(-2, 2)$. (If the real number s lies in the intervals $(-\infty, -2)$ and $(2, \infty)$ we can use the relation $G_{\mathrm{R}}(s) = G(s)$, with $G_{\mathrm{I}}(s) \equiv 0$.) In the first stage of the analysis the standard analytic continuation formula (Erdélyi *et al* 1953, p 109, equation (4))

$${}_2F_1\left(\frac{1}{8}, \frac{5}{8}; 1; \xi^2\right) = \frac{[\Gamma(\frac{1}{4})]^2}{2^{3/4}\pi^{3/2}}\xi^{-1/4}{}_2F_1\left(\frac{1}{8}, \frac{1}{8}; \frac{3}{4}; 1 - \frac{1}{\xi^2}\right) - \frac{2^{7/4}\pi^{1/2}}{[\Gamma(\frac{1}{4})]^2}\xi^{-7/4}(1-\xi^2)^{1/4}{}_2F_1\left(\frac{7}{8}, \frac{7}{8}; \frac{5}{4}; 1 - \frac{1}{\xi^2}\right) \tag{4.1}$$

is applied to equation (3.2), where $|\arg(\xi^2)| < \pi$ and $\xi = 2/t$. It is then possible to evaluate the limit defined in equation (1.6). Hence, we obtain the formula

$$G_{\mathrm{R}}(s) = \frac{[\Gamma(\frac{1}{4})]^4}{8\pi^3}\left(1 - \frac{s^2}{4}\right)^{-1/4}\left[{}_2F_1\left(\frac{1}{8}, \frac{1}{8}; \frac{3}{4}; 1 - \frac{s^2}{4}\right)\right]^2 - \frac{s}{\pi}{}_2F_1\left(\frac{1}{8}, \frac{1}{8}; \frac{3}{4}; 1 - \frac{s^2}{4}\right){}_2F_1\left(\frac{7}{8}, \frac{7}{8}; \frac{5}{4}; 1 - \frac{s^2}{4}\right) + \frac{\pi}{[\Gamma(\frac{1}{4})]^4}s^2\left(1 - \frac{s^2}{4}\right)^{1/4}\left[{}_2F_1\left(\frac{7}{8}, \frac{7}{8}; \frac{5}{4}; 1 - \frac{s^2}{4}\right)\right]^2, \tag{4.2}$$

provided that $0 < s < 2$. When $-2 < s < 0$ we can calculate $G_{\mathrm{R}}(s)$ using (4.2) and the symmetry relation

$$G_{\mathrm{R}}(-s) = -G_{\mathrm{R}}(s). \tag{4.3}$$

It is also found that

$$G_{\mathrm{I}}(s) = \frac{[\Gamma(\frac{1}{4})]^4}{8\pi^3}\left(1-\frac{s^2}{4}\right)^{-1/4}\left[{}_2F_1\left(\frac{1}{8},\frac{1}{8};\frac{3}{4};1-\frac{s^2}{4}\right)\right]^2 - \frac{\pi}{[\Gamma(\frac{1}{4})]^4}s^2\left(1-\frac{s^2}{4}\right)^{1/4}\left[{}_2F_1\left(\frac{7}{8},\frac{7}{8};\frac{5}{4};1-\frac{s^2}{4}\right)\right]^2, \tag{4.4}$$

where $-2 < s < 2$. Formulae (4.2) and (4.4) clearly show the singular behaviour of $G_{\mathrm{R}}(s)$ and $G_{\mathrm{I}}(s)$ respectively, as $s^2 \to 4-$.

The behaviour of $G_{\mathrm{R}}(s)$ and $G_{\mathrm{I}}(s)$ in the neighbourhood of $s = 0$ can be investigated by applying the alternative analytic continuation formula (Erdélyi *et al* 1953, p 108, equation (2))

$${}_2F_1\left(\frac{1}{8},\frac{5}{8};1;\xi^2\right) = \frac{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{(2\pi)^{3/2}}(-\xi^2)^{-1/8}{}_2F_1\left(\frac{1}{8},\frac{1}{8};\frac{1}{2};\frac{1}{\xi^2}\right) - \frac{2\pi^{\frac{1}{2}}}{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}(-\xi^2)^{-5/8}{}_2F_1\left(\frac{5}{8},\frac{5}{8};\frac{3}{2};\frac{1}{\xi^2}\right), \tag{4.5}$$

where $|\arg(-\xi^2)| < \pi$, to equation (3.2). Hence we find that

$$G_{\mathrm{R}}(s) = \frac{s}{\pi 2^{3/2}}\left(1-\frac{s^2}{4}\right)^{-1/4}{}_2F_1\left(\frac{1}{8},\frac{1}{8};\frac{1}{2};\frac{s^2}{4}\right){}_2F_1\left(\frac{5}{8},\frac{5}{8};\frac{3}{2};\frac{s^2}{4}\right) \tag{4.6}$$

and

$$G_{\mathrm{I}}(s) = \left(1-\frac{s^2}{4}\right)^{-1/4}\left\{\frac{[\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})]^2}{16\pi^3}\left[{}_2F_1\left(\frac{1}{8},\frac{1}{8};\frac{1}{2};\frac{s^2}{4}\right)\right]^2 - \frac{\pi}{2[\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})]^2}s^2\left[{}_2F_1\left(\frac{5}{8},\frac{5}{8};\frac{3}{2};\frac{s^2}{4}\right)\right]^2\right\}, \tag{4.7}$$

where $-2 < s < 2$. We see from these results that

$$G_{\mathrm{I}}(0) = \frac{[\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})]^2}{16\pi^3} = 0.642\,882\,248\,294\ldots, \tag{4.8}$$

with $G_{\mathrm{R}}(0) = 0$. Finally, we note that $G_{\mathrm{R}}(s)$, $G_{\mathrm{I}}(s)$ and the weight function $\rho(s)$ are all solutions of the basic differential equation (3.14) with $t = s$.

5. Concluding remarks

Formulae (4.4) and (4.7) for $G_{\mathrm{I}}(s)$ enable one to obtain closed-form expressions for the weight function $\rho(s)$ in equation (1.9). It is seen, therefore, that our results give an exact solution of the moment problem (see Shohat and Tamarkin 1943) associated with the set of moments $\{\mu_k; k = 0, 1, 2, \ldots\}$. We can use (1.8), (3.17) and (4.4) to express the moment integral (1.9) in the hypergeometric form

$$\frac{(\frac{1}{2})_n}{(1)_n}{}_3F_2\left[\begin{matrix}-n, & \frac{1}{4}, & \frac{3}{4}; & \\ & & & 1\\ 1, & 1; & & \end{matrix}\right] = \int_0^1 x^{2n}\left\{\frac{[\Gamma(\frac{1}{4})]^4}{2\pi^4}(1-x^2)^{-1/4}\left[{}_2F_1\left(\frac{1}{8},\frac{1}{8};\frac{3}{4};1-x^2\right)\right]^2 - \frac{16}{[\Gamma(\frac{1}{4})]^4}x^2(1-x^2)^{1/4}\left[{}_2F_1\left(\frac{7}{8},\frac{7}{8};\frac{5}{4};1-x^2\right)\right]^2\right\}\mathrm{d}x. \tag{5.1}$$

This result has been checked by evaluating the integral numerically for various values of $n = 0, 1, 2, \ldots$.

We also note that it is possible to derive a series representation for the integral (1.1) by substituting (3.17) into (1.4) and integrating term by term. It is found that

$$I(t) = \ln t - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{\left(\frac{1}{2}\right)_n}{(1)_n}\,{}_3F_2\left[\begin{matrix}-n, & \frac{1}{4}, & \frac{3}{4}; & \\ & & & 1\\ 1, & 1; & & \end{matrix}\right]\left(\frac{2}{t}\right)^{2n}, \tag{5.2}$$

where $|t| \geqslant 2$. When $t = 2$ we can also use (5.2) and (1.2) to obtain the summation formula

$$\sum_{n=1}^{\infty}\frac{1}{n}\frac{\left(\frac{1}{2}\right)_n}{(1)_n}\,{}_3F_2\left[\begin{matrix}-n, & \frac{1}{4}, & \frac{3}{4}; & \\ & & & 1\\ 1, & 1; & & \end{matrix}\right] = 8\left(\ln 2 - \frac{2}{\pi}\mathrm{G}\right). \tag{5.3}$$

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Appendix A.

In this appendix we give an alternative procedure for evaluating the contour integral

$$Q_j = \frac{(-4)^j}{2\pi\mathrm{i}}\int_{\mathcal{C}_1}\frac{(1+z)^j}{z^{1+2j}}\,{}_2F_1\left[\frac{1}{2},\frac{1}{2};1;-z(2+z)\right]\mathrm{d}z. \tag{A.1}$$

We begin by applying the inverse Landen transformation

$${}_2F_1\left[\frac{1}{2},\frac{1}{2};1;-z(2+z)\right] = \frac{2}{(2+z)}\,{}_2F_1\left[\frac{1}{2},\frac{1}{2};1;\frac{z^2}{(2+z)^2}\right] \tag{A.2}$$

and the standard integral representation

$${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right) = \frac{1}{\pi}\int_0^{\pi}\frac{1}{\sqrt{1-k^2\sin^2\theta}}\,\mathrm{d}\theta \tag{A.3}$$

to the integrand in (A.1). Hence we obtain

$$Q_j = \frac{(-4)^j}{\pi}\int_0^{\pi}\mathrm{d}\theta\,\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{C}_1}\frac{2(1+z)^j}{z^{1+2j}\sqrt{4+4z+z^2\cos^2\theta}}\,\mathrm{d}z. \tag{A.4}$$

Next we define an algebraic transformation function $y = y(\theta, z)$ which satisfies the quadratic equation

$$A(\theta)y^2 + B(\theta, z)y + C(\theta) = 0, \tag{A.5}$$

where

$$A(\theta) = (1-\cos\theta)^2, \tag{A.6}$$

$$C(\theta) = (1+\cos\theta)^2, \tag{A.7}$$

$$B(\theta, z) = A(\theta) + C(\theta) + 16[(1+z)/z^2]. \tag{A.8}$$

The function $y(\theta, z)$ clearly has two branches which are given by

$$y_\pm = y_\pm(\theta, z) = \frac{1}{2A(\theta)}\left[-B(\theta, z) \pm \frac{4}{z^2}(2+z)\sqrt{4+4z+z^2\cos^2\theta}\right]. \tag{A.9}$$

It follows from (A.9) that

$$A(\theta)y_\pm - \frac{C(\theta)}{y_\pm} = \pm\frac{4}{z^2}(2+z)\sqrt{4+4z+z^2\cos^2\theta}. \tag{A.10}$$

We also obtain the further relation

$$-\frac{16}{z^2}(1+z) = \left(1+\frac{1}{y_\pm}\right)\left[A(\theta)y_\pm + C(\theta)\right] \tag{A.11}$$

from (A.5). If equation (A.11) is differentiated with respect to z it is found that

$$\frac{16}{z^3}(2+z) = \frac{1}{y_\pm}\frac{\mathrm{d}y_\pm}{\mathrm{d}z}\left[A(\theta)y_\pm - \frac{C(\theta)}{y_\pm}\right]. \tag{A.12}$$

We can now use the single-valued function $y_+(\theta, z)$ and equations (A.10)–(A.12) to change the contour integration variable in (A.4) from z to y_+. This procedure yields

$$Q_j = \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta \frac{1}{4\pi \mathrm{i}}\int_{\mathcal{C}_2} \frac{1}{y_+}\left(1+\frac{1}{y_+}\right)^j [\sin^4(\theta/2)y_+ + \cos^4(\theta/2)]^j \,\mathrm{d}y_+, \tag{A.13}$$

where the contour $\mathcal{C}_2$ encircles the origin with a winding number $+2$. The application of the residue theorem to the contour integral in (A.13) leads to the formula

$$Q_j = \sum_{m=0}^{j}\binom{j}{m}^2 \frac{2}{\pi}\int_0^{\pi/2} \sin^{4m}\varphi \cos^{4j-4m}\varphi \,\mathrm{d}\varphi. \tag{A.14}$$

After integrating over the angle φ we find that

$$Q_j = \frac{1}{(2j)!}\sum_{m=0}^{j}\binom{j}{m}^2 \left(\frac{1}{2}\right)_{2m}\left(\frac{1}{2}\right)_{2j-2m}. \tag{A.15}$$

Finally, we carry out various manipulations of the Pochhammer symbols in order to express (A.15) in the hypergeometric form

$$Q_j = \frac{(\frac{1}{4})_j(\frac{3}{4})_j}{(\frac{1}{2})_j(1)_j}\,{}_4F_3\left[\begin{matrix} -j, & -j, & \frac{1}{4}, & \frac{3}{4}; \\ -j+\frac{1}{4}, & -j+\frac{3}{4}, & 1; & \end{matrix}\; 1\right]. \tag{A.16}$$

This result is in agreement with the earlier calculations given in section 2. The important feature of this alternative approach is that it does *not* use the hypergeometric identity (2.24).

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